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A Note on Tachyon Moduli and Closed Strings

Bruno Carneiro da Cunha[1]

Helsinki Institute of Physics
P.O. Box 64, FIN-00014 University of Helsinki, Finland

and

Departamento de Física, Universidade Federal de Pernambuco[2]
CEP 53901-970, Recife, Pernambuco, Brazil

Abstract

The collective behavior of the SL$(2,\mathbb{R})$ covariant brane states of non-critical $c = 1$ string theory, found in a previous work, is studied in the Fermi liquid approximation. It is found that such states mimick the coset WZW model, whereas only by further restrictions one recovers the double-scaling limit which was purported to be equivalent to closed string models. Another limit is proposed, inspired by the tachyon condensation ideas, where the spectrum is the same of two-dimensional string theory. We close by noting some strange connections between vacuum states of the theory in their different interpretations.

[1]bcunha@df.ufpe.br
[2]Permanent address

1 Some Remarks on Tachyon Condensation and 2-d String Theory

Two dimensional models of strings have always been helpful to describe various new phenomena in the theory. Discrete states, non-perturbative models and the open-closed string duality being only a few. There is however an inherent danger in taking symplified models given that the algebraic structures tend to reduce to the same form, even though they may arise from physically distinct properties.

The case at hand are the algebras related to $SL(2,\mathbb{R})$. The group is so ubiquitous in physics that sometimes an interpretation of its appearance becomes a daunting task. For instance, the relation of the loop algebra of $SL(2,\mathbb{R})$ and W_∞ has for long been known in the context of two-dimensional gravity. The latter algebra has a multitude of realizations, like for instance as the moments of fixed spin of U(1) currents in the two-dimensional free boson field [1]. Applications of the algebra structure to the double scaling limit of matrix models only furthers the problem given its robustness with respect to the choice of potentials. There is then the real danger that the lessons one takes from the symplified models are properties not based on principles that can be applied in more realistic theories.

In this letter we investigate the relationship between the collective behavior of unstable D0 branes and closed string backgrounds in two dimensional string theory. We will start by arguing that the collective behaviour is nothing more than the field theory of on-shell boundary states of a single boson, which with few extra assumptions can be shown to be equivalent to a gauged $SL(2,\mathbb{R})$ WZW model. The latter will be interpreted as Dijkgraaf *et al.* [2], as an exact background for closed strings. The exercise shows a clear way of relating open string collective states, in a suitable limit, to closed string backgrounds. This is carried out in the spirit of [3], among others.

The construction is in way very similar to the one discussed in [4]. This fact permits us to give a "moduli space" interpretation of the appearance of the $SL(2,\mathbb{R})$ symmetry discussed in [5], and also of its gauging. The semiclassical limit is given by a dense Fermi fluid of D0 branes, and we view it in light of the symmetry. We show how the model recovers the double scaling limit, by taking excitations in a suitable limit of a sector with fixed value of the quadratic Casimir of $SL(2,\mathbb{R})$. We also show that the low-lying spectrum of the same sector does have an equivalence to the spectrum of closed strings [6]. We conclude by remarking a strange similarity between the vacuum state of the coset model, with its "gravitational" or "closed string" characteristic and the vacuum state of the bosonized collective field of branes, which can be thought of as a scalar field in de Sitter spacetime.

2 Single Particle Configuration Space

It has long been known that the boundary states of a free boson at the self-dual radius have a SU(2) group structure (see, for instance, [7]). They can be reinterpreted as the conformally invariant states of a brane on which a string with coordinate given by the single boson ends. The crucial ingredient to the group structure are the generators:

$$J^{\pm} = \oint \frac{dz}{2\pi i} e^{\pm i\sqrt{2}X(z)}, \quad J^3 = \oint \frac{dz}{2\pi i} \frac{1}{\sqrt{2}} i\partial X(z) \tag{2.1}$$

which can be shown to have a well-defined action even at infinite radius. The algebra gives a correspondence between on-shell boundary states for the Euclidean theory and SU(2) group elements,

at least at the self-dual radius. To wit, they can all be seen as an SU(2) rotation of the Dirichlet state:

$$|B\rangle = e^{i\theta_a J^a}|D\rangle, \tag{2.2}$$

with for instance the Neumann state being given by $\theta_a J^a = \theta_1 J^1 + \theta_2 J^2 + \theta_3 J^3 = -\pi J^1$ in the notation of [7]. Because of the state-operator correspondence, the lesson is that one also has a group parametrization of the conformally invariant boundary interactions in this model. This in turn gives a mapping between the zero mode of the boson $x = X(0)$ and the value of the tachyon (boundary) interaction $T(x)$ and SU(2):

$$T(x) = \theta_+ e^{i(x-\theta_3)/\sqrt{2}} + \theta_- e^{-i(x-\theta_3)/\sqrt{2}}, \tag{2.3}$$

which associates a line of SU(2) elements to a given value of x and T, since one has the symmetry

$$\theta_\pm \to \theta_\pm e^{\pm\alpha/\sqrt{2}}, \quad \theta_3 \to \theta_3 + \alpha. \tag{2.4}$$

This symmetry will be crucial in the main course of this paper. Its space-time interpretation the two-dimensional gauge field, whose effect for a single brane is just to trivialize the action of the axial subgroup

$$g \to hgh, \quad h = e^{i\alpha J_3}. \tag{2.5}$$

In the following we will perform another abuse of the notation and call J_3 as the operator that translates x. The natural parametrization of SU(2) is then

$$g = e^{i(\phi+\alpha)J_3} e^{i\theta J_1} e^{-i(\phi-\alpha)J_3}. \tag{2.6}$$

The point of view taken in this paper is that the local algebra su(2) does provide a powerful tool for analysis of the condensation process. The basis of the argument will be outlined in the following paragraphs and sections. For now it is worth stressing the proposal is *not* the notorious exact correspondence between the free boson and the SU(2) WZW model at $k = 1$. The construction will be inspired by the McGreevy-Verlinde's [8] "fluid of branes" and as such we will be only interested in the zero modes of the embedding coordinates $\{\phi, \theta\}$, instead of the whole spetrum of the boson. Also, it is known that, while the SU(2) current (local) structure is pervasive, the global structure of the moduli space can be quite different [9], with global identifications, decompactifications and especial discrete states showing up depending on which orbifold of the uncompactified boson one takes. These effects can be accounted for by a suitable restriction on the allowed representations of the algebra. All of this considered, we will have the local structure of the boson at self-dual radius as paradigm: in this case the configuration space is $\mathrm{SU}(2) \simeq S^3$, which for the single particle is reduced to S^2 as the effect of fixing the symmetry (2.5).

As Sen argued in a series of papers, ([10, 11], see [12] for a review), the naive Wick rotation of the boson does provide a realistic picture of the tachyon condensation process in two dimensions, where oscillator modes are supressed by the BRST constraint. As far as the configuration space is concerned, the Wick rotation brings the group structure to $\mathrm{SL}(2,\mathbb{R})$. This fact was anticipated by Gaberdiel *et al.*, [13], where it is argued that if one does not bother about unitarity, the moduli space of the boundary states at the self-dual radius is just the complexified algebra generated by the currents (2.1), $\mathrm{SL}(2,\mathbb{C})$. Some of the states would have a strange interpretation in string theory: branes at imaginary positions, for instance. With the finding that these do actually correspond to closed string backgrounds [14], one is tempted to take the assumption that the Wick rotated

boson's conformal configuration space is in fact just a truncation of the same $\mathrm{SL}(2,\mathbb{C})$, with a reality condition suitable to study time-like bosons. For the Euclidianized theory, *i.e.*, a space-like boson, the truncation would yield the SU(2) current structure reviewed above. For the Lorentzian theory, *i. e.*, a time-like boson, the truncation yields $\mathrm{SL}(2,\mathbb{R})$ currents. In the latter case, the work of [13] raised suspicions about whether the generic states constructed in the Lorentzian case allow for a sensible Hilbert space structure. We will have more to say about this below.

In the Euclidian theory, the global structure in the self-dual point can be determined from the local algebra and properties of Ishibashii states. When one studies the behavior of the states under generic SU(2) transformations, one begins with the Neumann state $|N\rangle$ and applies to it finite SU(2) transformations. By considering the actual, effective, value of the tachyon that relates the Dirichlet to the Neumann state in (2.2), one sees that it is actually renormalized to a finite value [7]. Then, as far as boundary states are concerned, one can raise the value of the tachyon field "past infinity" to a whole different sector not available classically. In particular, a SU(2) rotation will bring the original Neumann state to a different one:

$$e^{2\pi i J_1}|N\rangle_{\mathrm{SU}(2)} = 2^{-\frac{1}{4}}\sum_{j,m} e^{2\pi i j}|j,m,-m\rangle\rangle, \tag{2.7}$$

which still satisfies the Neumann condition in the absence of Wilson lines $(J_3+\bar{J}_3)|N\rangle = 0$. There is a similar "doubling" of the Dirichlet state. These differentiations can be modelled in the proposed association by seeing the SU(2) element as an ordinary unitary 2×2 matrix. The group geometry allows one to see that the region covered by classical values of the tachyon field is only a coordinate patch of the full space of boundary perturbations. This feature is particular to the global SU(2) structure that shows up at the self-dual radius. In fact, it is the existence of Ishibashii states with half-integer j in (2.7) that makes for the non-triviality of the 2π rotation. By contrast, in type 0B string theory where one has instead the "real" SO(3) structure, there will be no doubling.

One can extend these two cases to the Lorentzian theory by performing a Wick rotation to the generator J_3, bringing the group structure to $\mathrm{SL}(2,\mathbb{R})$. The generator J_1 will now yield the elliptic subgroup of $\mathrm{SL}(2,\mathbb{R})$, U(1). To make the correspondence precise, we will take the Dirichlet state as a reference, where the group element associated to it will be the identity. Fixing $\alpha = 0$ and performing the Wick rotation $\phi = it$ in (2.6), one has

$$g = e^{-tJ_3}e^{2i\theta J_1}e^{tJ_3}, \tag{2.8}$$

whose values of t and θ parametrize the allowed, physically distinct one-brane configuration states. The J_3 current now is interpreted as the Hamiltonian, implementing t translations, and θ is related to the value of the tachyon, in a manner that $\theta = \pm\pi/2$ correspond to the Neumann states. Note that the latter are invariant by t translations, and then fixed points of the condensation process.

The condensation process is then viewed as the flow of J_3 in this configuration state. The geometrical action for this flow for a curve parametrized by s is

$$S_{\mathrm{single}} = -m\int ds\sqrt{-\mathrm{Tr}[(g^{-1}\dot{g})^2]} = -m\int ds\ \sqrt{-\dot{\theta}^2+\cos^2\theta\,\dot{t}^2}, \tag{2.9}$$

which, upon the change of coordinates $\cosh\tau = \sec\theta$ can be recognized as the DBI action, the worldline action for a single D-brane. Note that only the sector between $\theta = -\pi/2$ (corresponding to $|N'\rangle$) and $\theta = \pi/2$ ($|N\rangle$) is mapped through this transformation. Classically this poses no problem since the movement for finite t is restricted to this region.

The geometry in itself is quite interesting. The metric that arises from the parametrization above for the Euclidean case is that of a sphere $d\theta^2 + \cos^2\theta\, d\phi^2$, whereas in the Lorentz case we have a two-dimensional space with constant curvature:

$$ds^2 = d\theta^2 - \cos^2\theta\, dt^2 \tag{2.10}$$

which can be represented isometrically as the one-leaf hyperboloid in $\mathbb{R}^{2,1}$ (Fig. 1.)

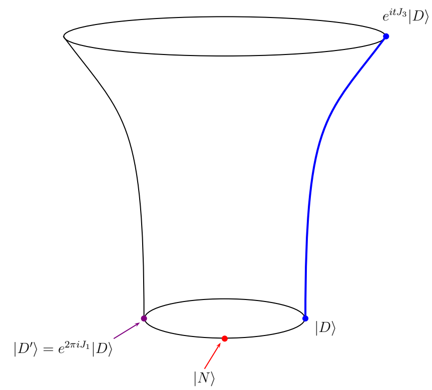

Figure 1: *The global geometry of the configuration space for a single brane, in the case where one has a Wick-rotated counterpart of the self-dual radius. One needs to act by* $e^{4\pi J_1}$ *for a full turn.*

In the Euclidian theory, one can interpret the "new" Dirichlet as being the antipode point to $|D\rangle$ in the sense that the former is labelled by $\theta = 0$ whereas the latter has $\theta = \pi$. In the Lorentzian theory this still holds, but one can define an "imaginary position" to $|D'\rangle$ via the metric (2.10). From the metric one finds the $\mathrm{SL}(2,\mathbb{R})$-invariant distance between two points p and p':

$$Z(p,p') = \cosh d(p,p') = \cosh(t-t')\cos\theta\cos\theta' - \sin\theta\sin\theta' \tag{2.11}$$

from which one sees that $|D'\rangle$ sits at the "imaginary distance" $d = i\pi$. Note that $d(p,p')$ is periodic in θ, and this periodicity will translate to an "imaginary periodicity" in t when one computes $\mathrm{SL}(2,\mathbb{R})$ invariant quantities as we will do in the next session. For instance, an $\mathrm{SL}(2,\mathbb{R})$-invariant propagator will display this periodicity in imaginary time, a thermal behavior. For type 0B the same argument as above yields half of the periodicity, as if $|D\rangle$ and $|D'\rangle$ were identified. The periodicity must be mimicked in models of tachyon condensation without the geometric picture. Tachyon insertions must be replaced by an array of insertions at imaginary positions to yield the closed string amplitudes [14].

One notes then that the $\mathrm{SL}(2,\mathbb{R})$ structure allows one to relate the radius of compactification in the Euclidianized theory to the radius of compactification of the variable θ above. Because of the local $\mathrm{SL}(2,\mathbb{R})$ structure, picking a different compactification of θ amounts to truncating the action of the universal covering group of the local structure, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. One is thus tempted to argue that the quantum mechanical states in generic tachyon condensation process can be obtained from some suitable truncation of the universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. We will give some support for this argument in the next section.

3 Dense Packing of Branes

In the Euclidean theory, the fact that the state is on-shell means that the quantities θ_a are constant throughout the renormalization group flow. The analogy with movements in space is clear at zero coupling: being heavy particles $m \propto 1/g$, the D0 branes are well-localized in the configuration space. In real time formalism, time evolution is implemented by the action of J_3. The condensation process is then the change of the "coordinates", or the values of fields, by the action of the time evolution operator. The Lagrangian obtained from this symmetry is the DBI action (2.9).

One would expect that the configuration space of a large number of branes at finite coupling to be completely different from the one described in the preceeding section, but surprisingly this is not the case. In the spirit of the preceeding section, let us associate a position in the space of allowed boundary conditions to each brane, that is, a $g_i \in \mathrm{SL}(2,\mathbb{R})$. The action of N of these branes will have a natural expansion in terms of N particle interactions:

$$S(\{g_i\}) = \sum_{i\neq j} \lambda_2 S_2(g_i, g_j) + \lambda_3 \sum_{i\neq j\neq k\neq i} S_3(g_i, g_j, g_k) + \dots \tag{3.12}$$

where the U(N) symmetry act as shuffling the indices. As such, the system has, along with the shuffling symmetry, the global $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ symmetry $g_i \to g g_i h$. The λ_i are coupling constants, which scale with the open string coupling $\lambda_n \propto g_o^n$. One can think of the S_2 term as measuring the "distance" in moduli space between brane "i" and "j". Note that now there is no *a priori* concept of distance between the indices, as opposed to usual matrix models. The action above is of course too general to be of any use but it can be reduced to an "universal form" through a few extra assumptions, or restrictions, on which brane configurations are considered.

The first and most important one is the assumption that the packing of branes will be *dense*, *i.e.*, that we can label the branes by indices i which take continuous values with the local topology of the target space, that is to say, $\mathbb{R}^3$. This is different from the usual matrix models in which the indices have the topology of the real line locally. Choosing this topology of $\mathbb{R}^3$ is natural from the moduli space point of view. Over these configurations, one can give a simple argument to symplify the intractable structure of (3.12). The relevant 2-point function has the structure:

$$\langle g_i | g_j \rangle_\lambda = D(g_i^{-1} g_j) + \lambda \sum_{g_k} D(g_i^{-1} g_k) D(g_k^{-1} g_j) + \dots = D_\lambda(g_i, g_j), \tag{3.13}$$

where $D(g)$ is the matrix representing the action of the group in the Ishibashii states. The sum will of course turn into an integral over all the intermediate configurations, but the relevant point is that whichever is the result of the integral, it will also be a class function of g_i and g_j, that is, it will be a function of $g_i^{-1} g_j$ of the trace type, $D_\lambda(g_i, g_j) = D_\lambda(g_i^{-1} g_j)$. Such functions can be written as a sum of representations of the group, and as such are just redefinitions of $D(g) \to D_\lambda(g)$. Since the particular representations which enter into $D(g)$ are not relevant to classical dynamics of the $g_i(s)$, we can then consider the first term only.

One has to be a bit more careful with the last statement. It is true that the form of the matrix $D_\lambda(g)$ does not matter for the dynamics of the zero modes, but one has to be careful to check whether non-unitary representations will be included in the expansion above. Like in the last section, we take the view of the tachyon condensation process as the dynamics of boundary states. In general, the latter are the Ishibashii states, which are superpositions of highest weight states of all the unitary representations of the current algebra. One is expected, then, to have only

unitary representations in the dynamics of the zero modes. This is extremely important since in the Lorentzian case the symmetry group is non-compact and hence unitarity could be broken. By this argument this situation will not arise.

Along with the density of the configurations, the other ingredient which will be relevant to this discussion is a well known result from matrix models. The large number of particles leads us to consider non-abelian tachyon fields, and the singlet sector of this field under the symmetry of exhange of indices can be represented by Fermi statistics on the single particle configuration space. One can then introduce a spinor operator $\psi(x_i)$ which populates an eigenvalue (D0-brane, or boundary term) of the particle i. The singlet sector will then be the "vacuum" state of this field operator, over which we will perturb the dynamics. Given that unitarity is preserved, spin-statistics tells us that this field should transform under the spinor representation of $\mathrm{SL}(2,\mathbb{R})$.

With the discussion above one sees that the type of correlation function we are interested are of the type:

$$\langle \psi(x_i) g(x_i)^{-1} g(x_j) \psi(x_j) \rangle \tag{3.14}$$

where $\psi(x_i)$ is the Fermi field that parametrizes the eigenvalues' distribution. The matrix $g(x_i)^{-1}g(x_j)$ parametrizes the overlap between the boundary states of the i^{th} and j^{th} eigenvalues. Now one sees why the details of the representations which take part of $D_\lambda(g)$ are not important: only the boundary state representative g and its action on the spinor, which must transform in the fundamental of $\mathrm{SL}(2,\mathbb{R})$ matter for the discussion. The effect of turning the coupling will be to change the fundamental representation to an equivalent one. We will omit from now on the indices i and j and assume a continuous distribution parametrized by three coordinates x. The argument above is reminiscent of the sewing techniques used in String Field Theory [16, 17].

We must also remember that the global configuration of the boundary states depends on the particular model of tachyon condensation we are taking. In the Euclidian case, for bosonic strings compactified at the self-dual radius this global structure is, $\mathrm{SU}(2)$. For type 0B it is $\mathrm{SO}(3) \approx \mathrm{SU}(2)/\mathbb{Z}_2$. As we argued in the last section, in general it will be some subgroup of the covering group of $\mathrm{SL}(2,\mathbb{C})$. In order to avoid these global complications, we will consider the invariant action based on a local connection of the gauge group:

$$S[\psi, \bar{\psi}, A_\mu] = k' \int d^3x \, \bar{\psi} \gamma^\mu (\partial_\mu + A_\mu) \psi, \tag{3.15}$$

where the condition for the existence of the map $g(x)$ which gives the boundary perturbation for each D-brane is translated to the flatness of the $\mathrm{sl}(2,\mathbb{R})$ connection:

$$\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]} = 0. \tag{3.16}$$

The solutions of (3.15) with flat connection are spinors of the type $\psi(x) = g(x)\eta$ with η constant transforming in the fundamental of $\mathrm{SL}(2,\mathbb{R})$. These configurations do indeed correspond to "occupied" localized states for all values of x. The extra terms – like gamma matrices – in the action are defined so that (3.15) is invariant under the choice of the Killing-Cartan form we pick for $\mathrm{SL}(2,\mathbb{R})$. In fact in this case a particular choice of the gamma matrices is nothing more than a choice for a basis of the Lie algebra. It will also be in our interest to keep the connection A_μ arbitrary, since the requirement (3.16) will turn out to be redundant.

If we integrate the fermions the resulting effective action for (3.15) will be the usual Chern-Simons, written in terms of the 1-form $A = A_\mu \, dx^\mu$:

$$S_{\mathrm{eff}} = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{3.17}$$

which represents the fact that the coordinates are just dumb indices whose metric structure cannot influence the dynamics. As promised above, the condition (3.16) is recovered by the equations of motion of (3.17). One can see the action above as a result of quantum fluctuations of the ψ field around the "vacuum" (singlet) state. In this case the 0-branes coalesce and a space-time, closed string metric structure arise under certain limits, as we will see in the remaining of the paper.

3.1 Of Gauge Choices and Constraints

The Chern-Simons term which dictates the dynamics of the map $g(x)$ has, as it is well-known, no local degrees of freedom. Non-trivial dynamics will arise if the domain of the map $g(x)$ has boundaries. This translates to the physical condition that our D0-branes cover only a finite-volume subset of the target-space. The reduction from (3.17) to the boundary is well studied (see, for instance, [4]). The effective action will be:

$$S = \frac{k}{2\pi} \int d^2z \; \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial} g) + \text{constraint}, \tag{3.18}$$

where the constraint's purpose is to gauge the symmetry (2.5). The reason behind the promotion of (2.5) from a global to a local symmetry is that one cannot physically control the value of the axial coordinate α chosen for all the D0-branes. By gauging the axial action, one removes the redundancy on the value of the tachyon field introduced by the boundary state group coordinates.

Let us further clarify the the form of the constraint. For the movement of a single particle, its effect is just the introduction of the Lagrange multipliers in the geometrical action:

$$S_{\text{single}} = \int ds \; \mathrm{Tr}(g^{-1}\dot{g})^2 + \lambda(s)\dot{\alpha}(s), \tag{3.19}$$

with g and α as in (2.6). One can alternatively think of the extra term as arising from the imposition of local invariance:

$$g(s) \to h(s)g(s)h(s), \tag{3.20}$$

with $h(s)$ as in (2.5), which has the advantage of not choosing coordinates *a priori*. This reproduces the constrained Lagrangean after one substitutes the equations of motion for the gauge field $A = h^{-1}\dot{h}$, or integrate it out in the path integral formalism. For the fluid, we would rather use the second formalism. If the real parameter u labels the branes at the boundary, one requires on (3.18) the invariance:

$$g(s,u) \to h(s,u)gh(s,u), \tag{3.21}$$

which introduces the following term in (3.18):

$$\text{constraint} = \mathrm{Tr}(g^{-1}\partial g A_{\bar{z}}) + \mathrm{Tr}(\partial g g^{-1} A_{\bar{z}}) + \text{c.c.} + \mathrm{Tr}(A_z A_{\bar{z}}). \tag{3.22}$$

To better connect this proposal with the literature, one can think of the action (3.18) as the bosonization of the DBI matrix model introduced in [5], or simply the effective action one has when the collective field derived from (2.9) is coupled to $\mathrm{SL}(2,\mathbb{R})$ currents.

$$T^i_a = \frac{i}{2}\left[\bar{\psi}\gamma^i\nabla_a\psi - \nabla_a\bar{\psi}\gamma^i\psi\right], \tag{3.23}$$

where i runs through the $\mathrm{SL}(2,\mathbb{R})$ indices. The equations of motion of (3.18) will then enforce the Ward identites of the collective field currents.

Fixing the coupling – or, alternatively, the mass of the brane – amounts to fix the value of the quadratic Casimir J^2. In the model above, this means that the three components of the currents will not be independent, since their sum of squares add to the value of the Casimir $m^2 = \frac{1}{4} + \mu^2$. Also, in order to rewrite the matrix model as the fermionic field, one has to impose the Gauss constraint of the gauge field. However, for finite k, this is not quite the exact thing to do [18]: in order to deal with the gauge symmetry one has to introduce ghosts and deal with the BRST quantization of (3.18). Luckily this particular program has been accomplished some time ago and we are left with the job of reinterpreting the spectrum.

3.2 $\mathrm{SL}(2,\mathbb{R})$ representations and branes.

Before analyzing the closed string excitations and the corresponding formulation in terms of the Liouville field, let us digress over the excitations of the theory above. The spectrum of the Euclidean theory will be then dictated by the representations of $\mathrm{SL}(2,\mathbb{R})$. In this case these are labelled by the quantum numbers $|j,m\rangle$, j a positive half integer and $|m| \leq j$, and whose wavefunctions are spherical harmonics in the θ, t plane. These correspond to the usual "special states" of the two dimensional string, *sans* the appropriate Liouville dressing, and an action equivalent to (3.17) has been proposed by Klebanov *et al.* [19]. We stress the appearance of the spinor degrees of freedom: from the fact that a single boundary perturbation transforms in the fundamental representation, one can associate the two independent solutions for the tachyon profile $e^{\pm iX(0)/\sqrt{2}}$ as the "spin up" and "spin down" states. One must also note that this analysis is again valid only insofar the "gauge fixing" $J_2 = 0$ can be consistently made, *i. e.*, in the large k limit. In the generic case one has to dress the states with the Liouville field as in the construction of the reference above. It is found that only the states with $m = j - 1$ survive the gauging process.

In the Lorentzian case, the situation changes somewhat. The sectors $J^2 = -(\frac{1}{4} + \mu^2)$ span the principal continuous series of $\mathrm{SL}(2,\mathbb{R})$. If we assume the compactified orbits of the elliptic generator J_3, the spectrum of each sector consists of states

$$\mathcal{H} = \left\{ |j = -\tfrac{1}{2} + i\mu, m + \nu\rangle \right\}, \quad m \in \mathbb{Z},\ \nu \in [-\tfrac{1}{2}, \tfrac{1}{2}) \tag{3.24}$$

One can think of the phase ν as parametrizing the twist of the boundary conditions on the matrix model – and therefore on the collective field. In [20] these where introduced to study the non-singlet sectors. From the open string perspective one would expect these to arise from compactification of the Euclidian boson (or array of branes at "imaginary positions") to non-rational radii (separation), and/or the turning of fluxes. For now let us consider the identity sector $\nu = 0$.

First let us consider the density of levels. The actual number of levels is of course actually infinite:

$$\mathcal{N}(\mu) = \sum_{m \in \mathbb{Z}} \langle j, m | j, m \rangle \tag{3.25}$$

but the sum can be regularized in an invariant way. For this consider the function

$$\mathcal{G}(\mu; t, \theta) = \sum_{m \in \mathbb{Z}} \langle j, m | D(g(t,\theta)) | j, m \rangle \tag{3.26}$$

where $g(t,\theta)$ is as in (2.6) and $D(g)$ is the representation of the $\mathrm{SL}(2,\mathbb{R})$ matrix g. One can then see that the character $\mathcal{G}(\mu; t, \theta)$ is a Green's function of the scalar Laplacian on the metric (2.10), and then it can be interpreted as a two-point function of a scalar field in that space, whose mass

is given by $M^2 = J^2 = \frac{1}{4} + \mu^2$. The two-point function at coincident points can be regularized by usual means [26] to yield:

$$\varrho(\mu) = \frac{1}{2\pi}\left(-\log\Lambda + \psi(-\tfrac{1}{2} + i\mu) + \psi(-\tfrac{1}{2} - i\mu)\right) \tag{3.27}$$

where $\psi(x)$ is the Euler digamma function. Comparing the expression above with the usual Matrix Model calculations, one finds readily that the UV cutoff introduced Λ has the interpretation as the *number* of branes (or the size of the matrices). In this formulation, the large N limit is exactly the same as the UV limit. Also, the parameter μ has the interpretation of the double scaled inverse coupling, and the expansion of (3.27) in inverse powers of μ show the characteristic $1/(2n)!$ behavior of closed strings [33]. It is interesting to point out the embedding of such field in the fermionic collective field model, by making use of the identity involving Jacobi functions

$$\cosh\frac{d}{2}\,\mathscr{D}^{-\frac{1}{2}+i\mu}_{-\frac{1}{2},-\frac{1}{2}}(\cosh d) + \sinh\frac{d}{2}\,\mathscr{D}^{-\frac{1}{2}+i\mu}_{-\frac{1}{2},\frac{1}{2}}(\cosh d) = \mathscr{D}^{-\frac{1}{2}+i\mu}_{0,0}(\cosh d). \tag{3.28}$$

The left hand side of the equation above can be seen to be the two spinorial components of the Green's function of a spinor field in the metric (2.10), whereas the left hand side is proportional to $\mathcal{G}(\mu; t, \theta)$. One can see that the bosonic excitations are seen as superpositions of two fermionic ones, with these two related by discrete symmetries. The same remarks where made in [23], although this makes clear that the end result of the condensation process should be stable, and the apparent instability of bosonic string should be an artifact of an artificial truncation of the spectrum by, for instance, "chopping off" the excitations with $\tau < 0$ and hoping that tunneling through the barrier will not happen.

This relationship will also be useful to gain insight on the model. Upon change of coordinates $\cosh\tau = 1/\cos\theta$ the equation for $\mathcal{G}(\mu; t, \tau)$ becomes

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \tau^2} + \frac{M^2}{\cosh^2\tau}\right)\mathcal{G} = \delta(t)\delta(\tau), \tag{3.29}$$

with $M = -J^2 = \frac{1}{4} + \mu^2$. So one can consider for the purposes of illustration an effective classical motion given by the ("tachyon") potential $V(\tau) = M/\cosh\tau$. In order to recover the classical limit one has to construct coherent states out of the spectrum. The starting point to this effect is the observation that we can use the states (3.24) to construct states localized in θ, or equivalently, localized in the tachyon value:

$$|j, \theta\rangle = \sum_{m\in\mathbb{Z}} e^{-im\theta}|j, m\rangle, \tag{3.30}$$

in which $j = -\frac{1}{2} + i\mu$ is a constant. These states span all the states available for a fluid moving in the metric (2.10). With these in mind one can construct delta-function normalized eigenstates of the Hamiltonian J_3, which is done in [30, 31]:

$$|j, \omega\rangle_+ = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\ \sqrt{1+\sin\theta}^{\,j-i\omega}\sqrt{1-\sin\theta}^{\,j+i\omega}|j, \theta\rangle. \tag{3.31}$$

Now we would like to draw the readers' attention to the "wavefunction"

$$\phi_\omega(\theta) = \frac{1}{\sqrt{2\pi}}\sqrt{1+\sin\theta}^{\,j-i\omega}\sqrt{1-\sin\theta}^{\,j+i\omega} \tag{3.32}$$

on which we would like to make two comments. Firstly, the probability distribution is *independent* of ω, $|\phi_\omega(\theta)|^2 = (\cos\theta)^{-1}$. By the change of variables $\cosh\tau = (\cos\theta)^{-1}$ one arrives at an expression equivalent to (3.31):

$$|j,\omega\rangle_+ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau \ e^{i\omega\tau} |j,\tau\rangle_+, \tag{3.33}$$

in which one redefines $|j,\tau\rangle$ to absorb the τ-dependent phase – $(\cosh\tau)^{-i\mu}$ – coming from the integral. Then one can represent a localized excitation (a D0 brane) as a minimum uncertainty wavepacket in the t,τ plane as represented in Fig. (2).

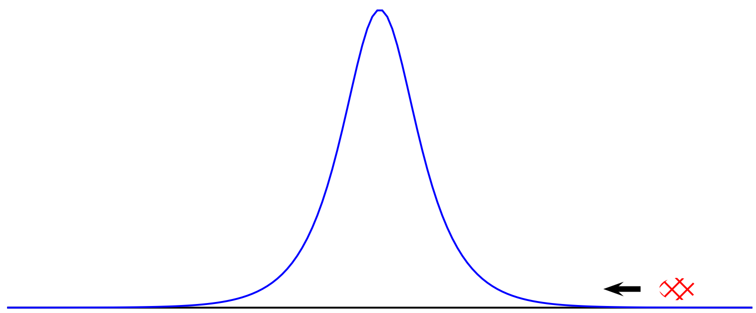

Figure 2: *A pictorial representation of a D0-brane type of excitation in the τ, t plane. The effective classical potential $M/\cosh\tau$ is also drawn.*

The second comment on (3.31) is about the fact that one does not need all of $|j,\theta\rangle$ to construct an eigenstate of J_3. In fact, only "half" of the circle suffices. This "double degeneracy" is a characteristic of the representation theory of $\mathrm{SL}(2,\mathbb{R})$ (see, for instance, [32]) and stems from the fact that the action of J_3 on the configuration space has two fixed points – the states $|N\rangle$ and $|N'\rangle$ in the language of section 2. By the action of J_3 states with $|\theta| < \pi/2$ are brought to states with $|\theta| < \pi/2$, and likewise for the other side of the circle, justifying the suffix "+" in (3.31) and in (3.33). One can at this point posits that, for the case of a configuration space given by $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, one has in fact to include an infinite number of these sectors. For our purposes, a single sector will suffice, since the dynamics of a single brane excitation is confined to it.

4 Closed string formalism

It is well-known that the action (3.18) has a direct interpretation in terms of closed strings moving in a particular background [2]. It is however less direct to see how exactly the closed string excitations result from the tachyon field. As stated above, the usual collective field formalism for the tachyon condensation process involves a gauge-fixing which is not exactly natural from the closed string point of view. They should be nonetheless equivalent as long as the number of particles N is large, and thus we barge ahead and try to model Liouville like excitations from the collective field formalism. To add to the confusing literature, we present two distinct ways of doing so.

4.1 The Non-relativistic (McGreevy-Verlinde) limit

As we saw in the preceeding section, after the gauge fixing, the tachyon process can be modelled by a relativistic particle moving in a sech potential. Since the $c = 1$ matrix model involves a inverted harmonic oscillator potential, it is natural to think that the double scaling limit will single out the maximum as the region of interest. As a matter of fact, the expansion of the potential $M/\cosh\tau$ will indeed yield an inverted harmonic potential. Borrowing from the literature, one considers the generating function for tachyon expectation values:

$$W(q,t) = \langle\Psi|e^{qJ_1}|\Psi\rangle, \tag{4.34}$$

more specifically, variations of W around states with definite energy $J_3 \approx \mu$. The quantity W is the quantum analogue of the inverted harmonic oscillator quantity given by:

$$W(\ell,t) \approx \int d\tau e^{\ell\tau}(p_+(\tau) - p_-(\tau)), \tag{4.35}$$

considered in the semiclassical limit, where expectation values are substituted by integrations over the Fermi sea. The boundary of the Fermi sea has momenta $p_\pm(\tau) = \pm\sqrt{2M\mu + M^2\tau^2/2}$, which enter into the semiclassical expression (4.35). The quantity W gives the "size of the universe" observable in Liouville theory. Since the time translation generator in (4.34) is given by J_3, one can compute the second derivative of W as in (4.34) with respect to time by using the identity:

$$[[e^{qJ_1}, J_3], J_3] = (-J_1\sinh q + J_2^2\sinh^2 q - \{J_2, J_3\}\sinh q(\cosh q - 1) + J_3^2(\cosh q - 1)^2)e^{qJ_1}, \tag{4.36}$$

Of which we perform a contraction:

$$J_{1,2} \to \sqrt{\rho}L_{1,2}, \quad q = \ell/\sqrt{\rho} \text{ and } \rho \to \infty. \tag{4.37}$$

Because $|\Psi\rangle$ has a definite value for J^2, one can then compute ${J_2}^2$ in terms of μ, q and J_3, the time derivative operator. Only the first and the second terms on the right hand side survive the contraction, and the result is

$$\frac{\partial^2}{\partial t^2}W = \ell\frac{\partial}{\partial\ell}W + \ell^2\frac{\partial^2}{\partial\ell^2}W - ((J_3)^2 - \mu^2)\frac{\ell^2}{\rho}W + \mathcal{O}(\rho^{-1}) \tag{4.38}$$

so, by substituting $J_3 = \mu - H$, with $H \ll \mu$ and taking $\mu = \rho$ one accomplishes the non-relativistic limit. The Wheeler-de Wit equation arises as one deforms the states $|\Psi\rangle$ by, say, adding an eigenstate of H [1].

In this picture the positive and negative energy solutions, as measured by the sign of H, give rise to opposing signs in last relevant term in (4.38). However, since we consider the Fermi sea to be filled up to $J_3 \approx \mu$, the right excitation is the absence of a negative energy state, which has the "right" sign for the usual correspondence between the Wheeler-de Witt equation and the Liouville field with a negative cosmological constant term [1]. The "hole" states thus give rise to another closed string sector described by another Liouville field which is independent in the weakly coupled regime $\mu \to \infty$. It would be interesting to go one order further in μ^{-1} to compute the mixing between the two sectors, but we will leave such discussion to future work.

4.1.1 Double scaling limit

It is perhaps worth pointing here that it is not clear that the double scaling limit will actually perform the process of contraction referred above. In fact, the computation of the density of levels done in the last section hints strongly that the couplings in (3.18) – particularly μ – are already scaled and hence there is no physical ground to the substitution of variables done in (4.37) other that it accomplishes the non-relativistic limit of the potential (3.29). Another argument against this comes from the density of levels of the flux backgrounds [34]. By changing the value of ν one selects amongst representations of $\mathrm{SL}(2,\mathbb{R})$. The density of levels is computed using the regularization of the character of the identity as above (3.27). The relevant matrix element involves the Jacobi functions [31]:

$$\varrho(\mu,\nu) = N_\nu \lim_{Z\to 1} \mathscr{D}_{\nu,\nu}^{-\frac{1}{2}+i\mu}(-Z) = \tilde{N}_\nu \;{}_2F_1(\tfrac{1}{2}-i\mu, \tfrac{1}{2}+2\nu+i\mu; 1+\nu, \tfrac{1}{2}(1-Z)) \tag{4.39}$$

where Z is as in [5] and ${}_2F_1$ is the usual hypergeometric function. Expanding the expression around $Z=1$ one finds the known expression involving the digamma function:

$$2\pi\varrho(\mu,\nu) = \frac{1}{\epsilon} + \operatorname{Re}\psi(\nu - \tfrac{1}{2} + i\mu) \tag{4.40}$$

which is exactly the one found in [34]. This infers that those representations with $\nu \neq 0$ model flux backgrounds, with the flux given by ν.

If the non-relativistic limit is to come from a sensible open-string picture, one has to compactify the tachyon direction to make the energy finite, as in [8]. One notes that such identification is done with respect to the tachyon value, here referred to as τ, and not the affine parameter of the Killing vector field, or the $\mathrm{SL}(2,\mathbb{R})$ isometry operator J_1, which is here called θ. Thus such identification is unnatural from the $\mathrm{SL}(2,\mathbb{R})$ perspective.

4.2 The ultra-relativistic (Sen) limit

The other way of constructing Liouville-like excitations is the well known coadjoint orbit reduction [4] of the $\mathrm{SL}(2,\mathbb{R})$ model. This can be understood geometrically as follows. Consider a particle sitting at the top of the potential (see Figure 2). Upon an $\mathrm{SL}(2,\mathbb{R})$ rotation, the time translation operator J_3 is transformed to:

$$e^{-i\theta J_1} J_3 e^{i\theta J_1} = \cosh\theta J_3 + i\sinh\theta J_2. \tag{4.41}$$

Now, as one takes the limit $\theta \to \infty$, the time translation symmetry is transformed to J_+, the generator of the elliptic subgroup of $\mathrm{SL}(2,\mathbb{R})$. By attributing a determined value for the "energy", *i. e.*, by restricting ourselves to the sector $J_+ = \sqrt{\mu}$, one accomplishes the reduction from the $\mathrm{SL}(2,\mathbb{R})$ model to Liouville [2]. One can understand the "boost" made above as the "pushing" of the Fermi sea to infinity in Figure 2. According to Sen, that is the minimum of the tachyon potential and where the closed string excitations are supposed to be localized. These excitations are exactly the "near-horizon" modes alluded to in [5].

Such excitations are modelled by states of constant $j = -\frac{1}{2} + i\mu$ in the "classical limit". Their dynamics can be recast in the form of a scalar field in a curved space, whose geometry is given by the geometry of the coset space (2.10). This is two-dimensional de Sitter, and hence all physical properties of the condensation process, and of the closed string excitations, can be interpreted

as properties of the propagating field in de Sitter. The first unsettling connection one makes is that the "black-hole" state in the coset model (of closed strings) has the interpretation of a "de Sitter" (*i. e.*, $\mathrm{SL}(2,\mathbb{R})$-invariant) vacuum state in the collective field of open strings. This shows an equivalence, at least in the two-dimensional model, between the black hole horizon and the cosmological horizon, the novelty being that the equivalence stems from the old open-closed string duality.

5 Discussion

In this article we discussed some semiclassical points pertaining to the condensation of D0 branes in non-critical string theory. It was argued how the spectrum of excitations should fit in the WZW coset model, with states in the former filling the representations of $\mathrm{SL}(2,\mathbb{R})$. We gave the interpretation of the condensation as the collective movement on the moduli space of allowed boundary states, and discussed two distinct limits where the closed string spectrum is recovered. Finally one posits that the black hole state constructed in the closed string sector should in fact represent a cosmological state constructed in the open string collective field.

One should note that this "geometrization" of the view of the condensation process is in fact necessary to avoid confusing coordinate transformations in field which only work in a patch of the configuration space. On the other hand true geometrization can only be achieved in the semiclassical limit and thus we will refrain from proposing a strict equivalence between the different points of view. One should remark, however, that the old mantra that all allowed boundary states correspond to closed string excitations is not followed in this work: the open string states corresponding to fluxes are gauge degrees of freedom and hence not dynamic. These states have to be dealt with by a gauge choice, which arises the question whether a global gauge fixing choice can be achieved. While the Coulomb gauge is usually used to yield Liouville theory in the double scaling limit, one is also tempted to not fix the gauge straight away and arrive through the symmetry argument presented here to the coset model. In the spirit of "covariance" the second method should be preferred.

What is true, nevertheless, is that one may see the spectrum of closed strings from the usual matrix model and the "near horizon" spectrum obtained here as different entities, arising from different limits, and gauge choices, from the picture of the dynamical variables in the boundary state formalism. One should note that the latter proposal implements Sen's ideas of how the closed degrees of freedom are encoded in the open string formalism and need no artificial constructions like an external thermal bath to keep the energy $\langle J_3 \rangle$ close to μ like in (4.38).

One is left wondering on what lessons can be transported to more realistic scenarios of tachyon condensation. Despite being a difficult guess to hazard, there is one redeeming feature of the study conducted here: the major pillar of sustentation of the work is the open-closed string duality. In our particular case this comes about as the T-duality that relates on the local group structure, Euclidianized to $\mathrm{SL}(2,\mathbb{R})$. Whether there are topological issues other than fluxes and broken symmetries that arise in the general problem is an open problem we would hope to return in the future.

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